From the foregoing I infer that, by substituting as above, any equation containing two variables, of the form $u = Ax^a + Bx^b + Cx^c + \&c.$, may be transformed to $\frac{du}{dx} = aAx^{a-1} + bBx^{b-1} + cCx^{c-1} + \&c.$

It may be said that other explanations, by different authors, perform the same transformations. They, however, first consider n to have some finite value which causes both u and x to change their values, but finally they assume $n = 0$, so that both u and x change back again to what they were at first; making two unnecessary changes as it seems to me.

My explanation supposes the values of u and x to remain the same during transformation and not to undergo any change at all; so that differentiation is simply Algebraic transformation; thus showing the connection between Algebra and the Calculus.

SOLUTION OF AN INDETERMINATE PROBLEM.

BY DAVID S. HART, A. M., M. D., STONINGTON, CONN.

[AT the conclusion of Dr. Matteson's solution, in Vol. II, page 49 of the ANALYST, we alluded to a request that had been made by Prof. Henkle, soon after the publication of Prof. Perkins' solution, that we publish the solution of a similar question that had been made by Dr. Hart some 20 years before, and promised to do so at some future time. As we have recently been reminded by Dr. Hart that his solution had not yet appeared in the ANALYST, we subjoin the solution alluded to, with the following prefatory remarks by Dr. Hart.—Ed.]

"This problem was I believe first given by Dr. J. R. Young in his Algebra, published a little more than 60 years ago. The answers he gives are small, but erroneous. Samuel Ward, in his American Edition of Young's Algebra, was the first to detect the error. It is probable that he went no farther than to satisfy himself that the answers must be very large numbers.

"It is now 26 years since I completed a general solution of this problem without conditions annexed. The condition that the squares shall be in arithmetical progression was solved by Prof. Kirkwood in Stoddard and Henkle's University Algebra, but his solution is a particular one involving very large numbers. Prof. Perkins has a general solution in the ANALYST, but it is limited to the case above mentioned. He finds the same numbers which I had already found, unknown to him, 25 years before.

"I sent my solution to Dr. Matteson 8 years ago, which he improved and extended so as to be still more general; and this solution, thus improved, was inserted in the ANALYST. Dr. M. ought to have given me credit for the general idea of solving all problems of this kind, in doing which I believe I was first in the field."

Problem.—Find three square numbers such, that if each be either increased or diminished by its root, the sums and differences shall be squares.

Let a^2x^2, b^2x^2, c^2x^2, be the numbers. Then

$$a^2x^2 \pm ax = \square, \qquad b^2x^2 \pm bx = \square, \qquad c^2x^2 \pm cx = \square.$$

Let $a^2x^2 + ax = m^2x^2$, also let $a^2x^2 - ax = m^2x^2$; then

$$x = \frac{a}{m^2 - a^2} \ldots (1); \text{ or } x = \frac{a}{a^2 - m^2}. \ldots\ldots\ldots (2)$$

Using the positive sign

$$b^2x^2 + bx = \frac{a^2b^2 - a^3b + abm^2}{(m^2 - a^2)^2} = \square, \; c^2x^2 + cx = \frac{a^2c^2 - a^3c + acm^2}{(m^2 - a^2)^2} = \square;$$

or, rejecting the square denominators,

$$a^2b^2 - a^3b + abm^2 = \square, \text{ and } \quad a^2c^2 - a^3c + acm^2 = \square,$$

both of which are squares if $m = a$. Let $m = a + n$; then, by substitution, these expressions become

$$abn^2 + 2a^2bn + a^2b^2 = \square, \text{ and } \quad acn^2 + 2a^2cn + a^2c^2 = \square.$$

Multiplying the first by c^2 and the second by b^2, we have

$$abc^2n^2 + 2a^2bc^2n + a^2b^2c^2 = \square = A^2, \; ab^2cn^2 + 2a^2b^2cn + a^2b^2c^2 = \square = B^2.$$

By subtraction we have

$$abc(c - b)n^2 + 2a^2bc(c - b)n = a(c - b)n \times (2abc + bcn) = A^2 - B^2$$
$$= (A + B)(A - B).$$

Now putting $a(c - b)n = A - B$, and $2abc + bcn = A + B$, and taking half the sum, we have $A = abc + \frac{1}{2}(ac + bc - ab)n$.

$$\therefore \; abc^2n^2 + 2a^2bc^2n + a^2b^2c^2 = A^2 = [abc + \tfrac{1}{2}(ac + bc - ab)n]^2.$$

Reducing this we obtain

$$n = \frac{4abc(ab + ac - bc)}{(ac + bc - ab)^2 - 4abc^2}; \; \therefore \; m = \frac{a[(ac + bc - ab)^2 - 4bc(bc - ab)]}{(ac + bc - ab)^2 - 4abc^2}.$$

Substituting this value of m in (1) and (2), above we finally obtain

$$x = \frac{\pm [(ac + bc - ab)^2 - 4abc^2]^2}{8abc(ac + bc - ab)(ac + ab - bc)(ac - ab - bc)}.$$

Multiplying by a, b, c, we have general expressions for ax, bx, and cx, the roots of the squares required.

As a, b, c, may be any numbers, they may be taken such that their squares shall be in arithmetical progression, geometrical progression, harmonical

progression; or that the sum of their squares shall be a square, or that they shall be the sides of a right-angled triangle. In fact the number of requirements is unlimited.

In the first case, let $a = 2rs - r^2 + s^2$, $b = r^2 + s^2$, $c = 2rs + r^2 - s^2$, which are general expressions for three square numbers in arithmetical progression; r, s, being any different numbers that will make a, b, c, all positive. Let $r = 2$, $s = 1$; then $a = 1$, $b = 5$, $c = 7$, and

$$x = \frac{151321}{7863240}. \quad \therefore \frac{151321}{7863240}, \frac{756605}{7863240}, \text{ and } \frac{1059247}{7863240}$$

are the roots for the positive sign in $a^2x^2 \pm ax = \square$, &c. In order to have a positive value of x to satisfy the negative sign, take $r = 4$, $s = 3$; then $a = 17$, $b = 25$, $c = 31$, and $x = \frac{(864571)^2}{11011044931800}$. Therefore

$$\frac{12707211238697}{11011044931800}, \frac{18687075351025}{11011044931800}, \frac{23171973435271}{11011044931800},$$

are the roots for the negative sign in $a^2x^2 \pm ax = \square$, &c.

SOLUTIONS OF PROBLEMS IN NUMBER 2.

101.—"From any point O, within the circumference of a circle, two lines are drawn making a constant angle with each other. These lines revolve about O in the plane of the circle, and from the points where they cut the circumference tangents are drawn. Find the locus of the intersection of these tangents.

SOLUTION BY PROF. W. W. BEMAN, ANN ARBOR, MICHIGAN.

LET a diameter through O represent the axis of X. Put $r =$ the radius of the given circle, $a =$ the distance of the point O from its centre, $\alpha =$ the constant angle included by the two given lines; x, y, coordinates of the intersection of the tangents, the origin being at the centre of the circle, and x', y', and x'', y'', coordinates of the points where the given lines cut the circumference of the given circle. Then we easily obtain:

$$xx' + yy' = r^2 \ldots . (1); \qquad xx'' + yy'' = r^2; \ldots\ldots (2)$$
$$x'^2 + y'^2 = r^2 \ldots . (3); \qquad x''^2 + y''^2 = r^2; \ldots\ldots (4)$$
$$(x' - x'')^2 + (y' - y'')^2 = (x' - a)^2 + y'^2 + (x'' - a)^2 + y''^2$$
$$- 2\sqrt{\{[(x' - a)^2 + y'^2][(x'' - a)^2 + y''^2]\}}\cos\alpha; \text{ or,}$$
$$x'x'' + y'y'' - a(x' + x'') + a^2 = \sqrt{[(r^2 + a^2 - 2ax')(r^2 + a^2 - 2ax'')]}\cos\alpha \ldots (5)$$

Combining (1) and (3),

$$x' = \frac{r}{x^2 + y^2}\left(rx \pm y\sqrt{(x^2 + y^2 - r^2)}\right); \quad y' = \frac{r}{x^2 + y^2}\left(ry \mp x\sqrt{(x^2 + y^2 - r^2)}\right).$$